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HYDRODYNAMICS AND ELECTROHYDRODYNAMICS OF ADIABATIC MULTIPHASE FLUIDS AND PLASMAS

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Abstract—Poisson brackets are presented for single-pressure ideal multiphase hydrodynamic and electrohydrodynamic models of fluids and plasmas. Stationary multiphase flows are shown to be critical points of the sum of the energy and additional conservation laws associated with the kernels of these Poisson brackets. The constraint of a common pressure for the phases is shown to preclude Lyapunov stability for these stationary flows.

1. INTRODUCTION

Multiphase flow involves interpenetration of various material species. Hydrodynamic models describing such systems raise many questions, such as well posedness and stability, which can be addressed in the context of a Hamiltonian formulation. Such a formulation is the purpose of this paper.

Practical models of multiphase flow are typically derived by taking averages with respect to time, space, or statistically over microscopic domains to obtain macroscopic fluid descriptions. This averaging procedure is a rather subtle process, which has been given detailed description in Ishii (1975) and Nigmatulin (1979), resulting in a by now standard, single-pressure fluid description of multiphase flow.

An open problem about the basic single-pressure fluid model without dissipation concerns its ill posedness: the model is not hyperbolic; in one dimension the system has complex characteristic eigenvalues, see e.g. Gidaspow *et al.* (1973). For a linear system, this would indicate ill posedness of the Cauchy problem, whose solutions would not depend continuously on the initial data for arbitrarily high wavenumber, as discussed in Lax (1957). For nonlinear systems of the same type as the single-pressure inviscid model, complex characteristic eigenvalues indicate nonexistence of a bounded integral of the solution, as shown in Lax (1980).

The problem of ill posedness has been avoided in practice by introducing either viscous dissipation as in Arai (1980) and Stewart (1979), or additional pressures arising from surface tension as in Ramshaw & Trapp (1978), bubble inertia as in Bedford & Drumheller (1978), or other interfacial pressure jumps determined phenomenologically, see e.g. Ransom & Hicks (1984). Additional models and methods in two-phase flow are reviewed in Stewart & Wendroff (1984).

Despite the recent progress mentioned above, the theoretical situation concerning ill posedness of the basic nondissipative model is still unsatisfactory and its mathematical structure needs clarification. In this paper, we provide a Hamiltonian formalism for the basic single-pressure model and analyze within this context how Lyapunov stability is prevented, in comparison with single-species compressible fluids and multispecies, multi-pressure models.

Multiphase barotropic and adiabatic flows of uncharged materials at a single, common pressure are considered in sections 2 and 3, respectively. These standard theories are extended to a multiphase plasma of charged interpenetrating materials in section 4. In each case, we present a Hamiltonian formulation and identify conservation laws associated with existence of a nontrivial kernel of the corresponding Poisson bracket.

For single-phase flows (Holm *et al.* 1983) and for charged multifluid flows with multiple pressures (Holm 1984; Holm *et al.* 1985), the Hamiltonian formulation leads to sufficient criteria for Lyapunov stability in the neighborhood of steady solutions. However, because multiphase flows take place at a single, common pressure, the implicit dependence of the pressure on the entire set of macroscopic densities prevents the establishment of even linearized Lyapunov stability, as is illustrated in section 5 for multiphase barotropic flow in two dimensions. This result is consistent with the ill posedness of the multiphase equations in one dimension and its consequent sensitive dependence to high wavenumber perturbations in initial conditions discussed in Lax (1957; 1980), Ransom & Hicks (1984), and Stewart & Wendroff (1984).

Thus, the Hamiltonian procedure of stability analysis provides a framework in which it becomes clear that stability is prevented by the constraint of a common pressure for each phase. This Hamiltonian formulation also provides a useful springboard for amendments and generalizations. In the companion paper (Holm & Kupershmidt 1986) we extend the Hamiltonian formulation presented here for the standard model to derive a new, well-posed hyperbolic multiphase model whose steady equilibria are Lyapunov stable under certain conditions.

2 MULTIPHASE BAROTROPIC HYDRODYNAMICS

A local description of ideal, barotropic, multiphase flow is given in terms of the following variables as functions of space coordinates $\mathbf{x}$ with components x_i, $i = 1, 2, \cdots, n$ and time t:

- θ^s the volume fraction of material s in a unit volume; $s = 1, 2, \cdots, N$, $\sum_{s=1}^{N} \theta^s = 1$
- ρ^s the microscopic density of material s.
- $\bar{\rho}^s$ the macroscopic density of material s; $\bar{\rho}^s = \rho^s\theta^s$ (no sum).
- $\mathbf{v}^s$ the velocity of material s.
- P^s the pressure within material s, which is taken to be the same for all materials; $P^s = P$ for all s, in the standard, single-pressure theories.
- e^s the internal energy per unit mass of material s; $e^s = e^s(\rho^s)$ is the equation of state, so that $\rho^s = \rho^s(P)$ and $de^s = (P/(\rho^s)^2)d\rho^s$ for the barotropic, single-pressure case

The N constraints

$$\sum_{s=1}^{N} \theta^s = 1, \quad P^s(\rho^s) = P \quad \text{for all } s = 1, 2, \cdots, N$$

impose implicit dependences

$$P = P(\{\bar{\rho}^s\}), \quad \theta^s = \theta^s(\{\bar{\rho}^s\}),$$

since $\rho^s = \rho^s(P) = \dfrac{\bar{\rho}^s}{\theta^s}$, where $\rho^s(P)$ can be considered as given functions. As an illustration, consider the case of two species, $s = 1, 2$. Then, with $\theta^1 = \theta$, $\theta^2 = 1 - \theta$,

$$\bar{\rho}^{-1} = \theta\rho^1(P), \quad \bar{\rho}^2 = (1-\theta)\rho^2(P) = \left(1 - \frac{\bar{\rho}^1}{\rho^1(P)}\right)\rho^2(P),$$

which implies that $P = P(\bar{\rho}^1, \bar{\rho}^2) = P(\{\bar{\rho}^s\})$, and likewise, $\theta^s = \theta^s(\{\bar{\rho}^s\})$. Consequently, we consider P, θ^s to be given functions of $\{\bar{\rho}^s\}$.

An interesting thermodynamic consequence of this dependence $P(\{\bar{\rho}^s\})$ of the single pressure on the entire set of macroscopic densities is the "macroscopic sound speed" relation

$$(\bar{C}^s)^2 = \frac{\partial P}{\partial \rho^s} = \frac{P/\rho^s(P)}{\sum_\beta \bar{\rho}^\beta \partial e^\beta/\partial P}, \qquad [1]$$

which follows from substitution of the relation $\rho^s = \bar{\rho}^s/\theta^s$ into the thermodynamic first law, $de^s = (P/(\rho^s)^2)d\rho^s$. Thus,

$$\frac{\partial e^s}{\partial \bar{\rho}^\alpha} = \frac{P}{\rho^s\bar{\rho}^s}\delta^{s\alpha} - \frac{P}{\bar{\rho}^s}\frac{\partial \theta^s}{\partial \bar{\rho}^\alpha},$$

which leads to formula [1] via

$$\frac{\partial}{\partial \bar{\rho}^\alpha}\sum_\beta \bar{\rho}^\beta e^\beta = e^\alpha + \frac{P}{\rho^\alpha}, \qquad [2]$$

upon using $\Sigma_\beta \theta^\beta = 1$. Relations [1] and [2] will be found useful for casting the ideal multiphase equations into a Hamiltonian form and studying their stability properties.

The equations of ideal multiphase flow are (see e.g. Stewart & Wendroff 1984)

$$\partial_t \bar{\rho}^s + \operatorname{div} \bar{\rho}^s \mathbf{v}^s = 0, \qquad [3]$$

$$\partial_t v_i^s + v_j^s v_{i,j}^s = -\frac{\theta^s}{\bar{\rho}^s}P_{,i} - \phi_{,i}, \qquad [4]$$

where summation on repeated subscripts is implied (no summation convention is imposed on superscript s) and $\phi(\mathbf{x})$ is the potential for an external body force. Equation [3] expresses conservation of mass, while [4] determines the motion for each species with drag terms between constituents neglected. Using [3], an alternative form of the motion [4] is obtained:

$$\partial_t(\bar{\rho}^s v_i^s) + (\bar{\rho}^s v_i^s v_j^s)_{,j} = -\theta^s P_{,i} - \bar{\rho}^s \phi_{,i},$$

so that, in terms of the species momentum density $\mathbf{M}^s = \bar{\rho}^s \mathbf{v}^s$, we have

$$\partial_t M_i^s + \left(\frac{M_i^s M_j^s}{\bar{\rho}^s}\right)_{,j} = -\theta^s P_{,i} - \bar{\rho}^s \phi_{,i}. \qquad [5]$$

Energy conservation is a consequence of [3] and [4]. Namely,

$$\partial_t\left[\bar{\rho}^s\left(\frac{1}{2}|\mathbf{v}^s|^2 + e^s + \phi\right)\right] = -\operatorname{div}\left(\bar{\rho}^s\mathbf{v}^s\left[\frac{1}{2}|\mathbf{v}^s|^2 + e^s + \phi + \frac{\theta^s P}{\bar{\rho}^s}\right]\right) - P\frac{\partial \theta^s}{\partial t}.$$

Consequently, upon summation on s and use of $\Sigma_s \theta^s = 1$, the following quantity is seen to be conserved for species velocities $\mathbf{v}_s$ tangential to the boundaries of the domain of flow,

$$E = \sum_s \int \bar{\rho}^s\left(\frac{1}{2}|\mathbf{v}^s|^2 + e^s + \phi\right)d^n x, \qquad [6]$$

where $d^n x$ is the n-dimensional volume element. Throughout, the dimension of the volume element indicates which results are general, and which only apply in certain dimensions. The total energy of the system is conserved, provided the velocities $\mathbf{v}^s$ are tangent to the boundary. Likewise, summation over species of the momentum equations [5] implies conservation of those components of the total linear momentum which correspond to directions in which ϕ is translation invariant.

Another consequence of [3] and [4] is advection of each component of the specific vorticity $\boldsymbol{\omega}^s/\bar{\rho}^s = (\text{curl } \mathbf{v}^s)(\bar{\rho}^s)^{-1}$ for each species. Using the relation $\rho^s = \rho^s(P)$, one finds from [4] that, with $\boldsymbol{\omega}^s = \text{curl } \mathbf{v}^s$,

$$\partial_t \mathbf{v}^s = \mathbf{v}^s \times \boldsymbol{\omega}^s - \nabla\left(\frac{1}{2}|\mathbf{v}^s|^2 + h^s(P) + \phi\right), \qquad [7]$$

where $h^s(P)$ is the specific enthalpy, satisfying

$$\nabla h^s(P) = \frac{1}{\rho^s(P)} \nabla P .$$

Thus, by taking the curl of [7], we get

$$\partial_t \boldsymbol{\omega}^s = \text{curl}(\mathbf{v}^s \times \boldsymbol{\omega}^s), \qquad [8]$$

or, upon using [3] and expanding [8] one finds

$$(\partial_t + \mathbf{v}^s \cdot \nabla)\frac{\boldsymbol{\omega}^s}{\bar{\rho}^s} = (\bar{\rho}^s)^{-1}(\mathbf{w}^s \cdot \nabla)\mathbf{v}^s . \qquad [9]$$

Equation [8] as written is true for a barotropic fluid in three dimensions. In n dimensions, the corresponding statement is that the 2-forms $\mathrm{d}v^s = \mathrm{d}(v^s_i \mathrm{d}x^i) = v^s_{i,j}\mathrm{d}x^j \wedge \mathrm{d}x^i$ are "frozen in", i.e. $(\partial_t + \mathscr{L}\mathbf{v}^s)\mathrm{d}v^s = 0$, where $\mathscr{L}\mathbf{v}^s$ means Lie derivative with respect to the vector field $\mathbf{v}^s$ in standard notation, see e.g. Schutz (1980). In terms of components, $\omega^s_{ij} = v^s_{i,j} - v^s_{j,i}$, we find that

$$(\partial_t + \mathscr{L}\mathbf{v}^s)\mathrm{d}v^s = \frac{1}{2}[\partial_t\, \omega^s_{ij} + (\omega^s_{ij,k} - \omega^s_{ki}\partial_j - \omega^s_{jk}\partial_i)v^s_k]\mathrm{d}x^j \wedge \mathrm{d}x^i = 0 , \qquad [10]$$

which becomes [8] upon identifying $-\omega^s_{12} = \omega^s_3$, etc. in three dimensions.

Hamiltonian formalism. The main result of this section is that the equations of multiphase dynamics [3] and [5] can be written in the Hamiltonian form $\partial_t F = \{H,F\}$ with Poisson bracket $\{\ ,\ \}$ given for arbitrary functionals J, K of $\bar{\rho}^s$, $\mathbf{M}^s$ by

$$\{K,J\} = -\sum_s \int \mathrm{d}^n x \left\{\frac{\delta J}{\delta \bar{\rho}^s}\partial_j \bar{\rho}^s \frac{\delta K}{\delta M^s_j} + \frac{\delta J}{\delta M^s_i}\left[\bar{\rho}^s \partial_i \frac{\delta K}{\delta \bar{\rho}^s} + (\partial_j M^s_i + M^s_j \partial_i)\frac{\delta K}{\delta M^s_j}\right]\right\}, \qquad [11]$$

and Hamiltonian $H[\bar{\rho}^s, \mathbf{M}^s]$ given by

$$H = \sum_s \int \mathrm{d}^n x \left[\frac{|\mathbf{M}^s|^2}{2\bar{\rho}^s} + \bar{\rho}^s e^s + \bar{\rho}^s \phi\right], \qquad [12]$$

which, of course, is the total energy [6].

This Hamiltonian formulation can be verified directly, by using the variational derivatives

$$\frac{\delta H}{\delta \mathbf{M}^s} = \mathbf{v}^s, \quad \frac{\delta H}{\delta \bar{\rho}^s} = -\frac{1}{2}|\mathbf{v}^s|^2 + e^s + \frac{P}{\rho^s} + \phi ,$$

where we have used relation [2]. We find, letting $J = \bar{\rho}^s$, $K = H$ in [11], that [3] is verified;

$$\partial_t \bar{\rho}^s = \{H, \bar{\rho}^s\} = -\partial_j(\bar{\rho}^s v^s_j) .$$

Letting $J = M^s_j$, $K = H$ in [11], we find that [5] is verified as well;

$$\partial_t M^s_i = \{H,M^s_i\} = -(M^s_i v^s_j)_{,j} - \bar{\rho}^s \phi_{,i} - \bar{\rho}^s \partial_i\left(e^s + \frac{P}{\rho^s}\right) = -(M^s_i v^s_j)_{,j} - \bar{\rho}^s \phi_{,i} - \theta^s P_{,i}\,,$$

thanks to the first law $de^s = -P d(1/\rho^s)$.

The bracket [11] is a sum over species of N copies of the Poisson bracket for single-species, compressible flow which is given together with its mathematical interpretation in Holm & Kupershmidt (1983) and was introduced in Iwinski & Turski (1976), Dzyaloshinsky & Volovik (1980), and Morrison & Greene (1980). Thus, the Poisson bracket [11] for ideal barotropic multiphase flow has the same form as the Poisson bracket for single-phase flow, when expressed in terms of the macroscopic mass densities $\bar{\rho}^s$ and momentum densities $\mathbf{M}^s = \bar{\rho}^s \mathbf{v}^s$.

3 MULTIPHASE ADIABATIC HYDRODYNAMICS

For multiphase adiabatic flow, each specific internal energy e^s depends on both microscopic mass density ρ^s and specific entropy η^s through the equation of state $e^s = e^s(\rho^s,\eta^s)$. In this case, the first law becomes

$$de^s = T^s d\eta^s - P d\left(\frac{1}{\rho^s}\right), \qquad [13]$$

where T^s is the temperature of species s. The constraints $\Sigma_s \theta^s = 1$ and $P^s(\rho^s,\eta^s) = P$, $s = 1, \cdots, N$, impose functional dependences

$$P = P(\{\bar{\rho}^s\},\{\eta^s\}),\ \ \theta^s = \theta^s(\{\bar{\rho}^s\},\{\eta^s\}), \qquad [14]$$

for given functional relations $\rho^s = \rho^s(P, \eta^s) = \bar{\rho}^s/\theta^s$. The expressions [14] for P, θ^s are assumed to be known. As a consequence of [14], one finds from [13] that

$$\frac{\partial e^s}{\partial \bar{\rho}^a} = \frac{P}{\rho^s \bar{\rho}^s}\delta^{sa} - \frac{P}{\bar{\rho}^s}\frac{\partial \theta^s}{\partial \bar{\rho}^a},\ \ \frac{\partial e^s}{\partial \eta^a} = T^s \delta^{sa} - \frac{P}{\bar{\rho}^s}\frac{\partial \theta^s}{\partial \eta^a}\,. \qquad [15]$$

These expressions lead to the thermodynamic identities,

$$\frac{\partial}{\partial \bar{\rho}^a}\sum_s \bar{\rho}^s e^s = e^a + \frac{P}{\rho^a}, \qquad [16]$$

$$\frac{\partial}{\partial \eta^a}\sum_s \bar{\rho}^s e^s = \bar{\rho}^a T^a, \qquad [17]$$

upon using $\Sigma_s \theta^s = 1$. Identities [16] and [17] will be useful for extending the Hamiltonian formulation obtained in the previous section to the case of adiabatic flow. For multiphase adiabatic flow, the equations of motion are

$$\partial_t \bar{\rho}^s + \operatorname{div} \bar{\rho}^s \mathbf{v}^s = 0\,, \qquad [18a]$$

$$\partial_t \eta^s + \mathbf{v}^s \cdot \nabla \eta^s = 0\,, \qquad [18b]$$

$$\partial_t M^s_i + \left(\frac{M^s_i M^s_j}{\bar{\rho}^s}\right)_{,j} = -\theta^s P_{,i} - \bar{\rho}^s \phi_{,i}\,, \qquad [18c]$$

where the adiabatic condition requires that specific entropy η^s advects with the flow of each species. (Note that those flows are not isentropic, i.e. not constant entropy; rather they are adiabatic, i.e. no heat is exchanged across flow lines.)

A consequence of these equations analogous to [9] is, in three dimensions,

$$(\partial_t + \mathbf{v}^s \cdot \nabla)\Omega^s = 0\,, \tag{19}$$

where Ω^s, given by

$$\Omega^s = (\bar{\rho}^s)^{-1}\boldsymbol{\omega}^s \cdot \nabla \eta^s\,, \tag{20}$$

is the potential vorticity for material s which is transported by the flow of that species Conserved quantities associated with η^s and Ω^s are

$$F_{\Phi^s} = \int \mathrm{d}^3x\, \bar{\rho}^s\, \Phi^s(\eta^s, \Omega^s)\,, \tag{21}$$

for arbitrary functions Φ^s of two real variables for which the integrals F_{Φ^s} exist. Proof that the quantities F_{Φ^s} are conserved for velocities tangent to the boundary follows by direct computation using [18a], [18b], and [19], and integrating by parts. In n dimensions, the geometrical statement corresponding to [19] is that the 3-form $\mathrm{d}v^s\wedge\mathrm{d}\eta^s = -v^s_{i,j}\eta^s_{,k}\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$ is frozen into the flow of material s, i.e. $(\partial_t + \mathscr{L}\mathbf{v}^s)(\mathrm{d}v^s\wedge\mathrm{d}\eta^s) = 0$, where $\mathscr{L}\mathbf{v}^s$ is the Lie derivative and $\wedge$ denotes exterior product of differential forms. Formulation of [19] in terms of Lie derivatives shows its geometrical meaning. The proof follows by taking the exterior derivative of the motion equation written as

$$(\partial_t + \mathscr{L}\mathbf{v}^s)v^s + \frac{1}{\rho^s}\,\mathrm{d}P + \mathrm{d}\left(\phi - \frac{|\mathbf{v}^s|^2}{2}\right) = 0\,,$$

where $v^s = v^s_i\mathrm{d}x^i$. Upon using $[\mathrm{d}, \mathscr{L}\mathbf{v}^s] = 0$ and $\mathrm{d}^2 = 0$ (see e.g. Schutz 1980), the result of the exterior derivative is

$$(\partial_t + \mathscr{L}\mathbf{v}^s)\mathrm{d}v^s = \mathrm{d}P \wedge \mathrm{d}\left(\frac{1}{\rho^s}\right).$$

Then the adiabatic equations $(\partial_t + \mathscr{L}\mathbf{v}^s)\eta^s = 0$ imply that

$$(\partial_t + \mathscr{L}\mathbf{v}^s)(\mathrm{d}v^s\wedge\mathrm{d}\eta^s) = \mathrm{d}P\wedge\mathrm{d}\left(\frac{1}{\rho^s}\right)\wedge\mathrm{d}\eta^s = 0\,, \tag{22}$$

where the last equality is a consequence of the functional relation $\rho^s = \rho^s(P, \eta^s)$. In three dimensions [22] becomes

$$\partial_t(\boldsymbol{\omega}^s\cdot\nabla\eta^s) + \mathrm{div}\,[\mathbf{v}^s(\boldsymbol{\omega}^s\cdot\nabla\eta^s)] = 0\,,$$

and [19] follows upon combining this with the continuity equation for $\bar{\rho}^s$.

Hamiltonian formalism. Equations [18] of adiabatic multiphase hydrodynamics can be expressed in Hamiltonian form $\partial_t F = \{H,F\}$, with Poisson bracket given on the space of dynamical variables $\{\bar{\rho}^s,\eta^s,\mathbf{M}^s\}$ by

$$\begin{aligned}\{K,J\} = -\sum_s \int \mathrm{d}^n x \Bigg\{ &\frac{\delta J}{\delta\bar{\rho}^s}\,\partial_j\bar{\rho}^s\,\frac{\delta K}{\delta M^s_j} + \frac{\delta J}{\delta\eta^s}\,\eta^s_{,j}\,\frac{\delta K}{\delta M^s_j} \\ &+ \frac{\delta J}{\delta M^s_i}\left[\bar{\rho}^s\partial_i\,\frac{\delta K}{\delta\bar{\rho}^s} - \eta^s_{,i}\,\frac{\delta K}{\delta\eta^s} + \left(\partial_j M^s_i + M^s_j\partial_i\right)\frac{\delta K}{\delta M^s_j}\right]\Bigg\}\end{aligned} \tag{23}$$

and Hamiltonian $H = \Sigma_s H^s$,

$$H = \sum_s \int \mathrm{d}^n x \left[\frac{|\mathbf{M}^s|^2}{2\bar{\rho}^s} + \bar{\rho}^s e^s(\rho^s,\eta^s) + \bar{\rho}^s\phi\right]. \tag{23'}$$

The proof proceeds as in section 1 by substituting into the Poisson bracket [23] the variational derivatives

$$\frac{\delta H}{\delta \mathbf{M}^s} = \mathbf{v}^s ,$$

$$\frac{\delta H}{\delta \bar{\rho}^s} = -\frac{1}{2}|\mathbf{v}^s|^2 + e^s + \frac{P}{\rho^s} + \phi ,$$

$$\frac{\delta H}{\delta \eta^s} = \bar{\rho}^s T^s ,$$

obtainable using the thermodynamic relations [16] and [17].

The conserved quantities [21] lie in the kernel of the Poisson bracket [23] in three dimensions (that is, they Poisson commute with every functional of the variables $\{\bar{\rho}^s, \eta^s, \mathbf{M}^s\}$) so they will be conserved for any choice of Hamiltonian. See Holm & Kupershmidt (1983), for a mathematical interpretation of the Poisson bracket [23].

4 ADIABATIC MULTIPHASE ELECTROHYDRODYNAMICS

Adiabatic multiphase flow of ideal charged fluids creates a current. This current induces an electromagnetic field, which self-consistently influences the fluid motion through the Lorentz force. In this section, we will show how the self-consistent inclusion of an electromagnetic field alters the Hamiltonian structure for electrically charged adiabatic multiphase flows. Let a^s be the charge-to-mass ratio for material s, E_i the electric field, and A_i the magnetic vector potential, related to the magnetic field tensor B_{ij} by

$$B_{ij} = A_{i,j} - A_{j,i}.$$

The coupled electromagnetic and fluid equations consist of: dynamical Maxwell equations for the electromagnetic fields, conservation laws for mass and entropy of each species, and the motion equations for the fluid velocities. The sources for the electromagnetic fields are determined by the products of the parameters a^s (which could be zero for some species) with the macroscopic mass densities $\bar{\rho}^s$. The multiphase plasma (MPP) equations are, upon choosing the radiation gauge ($\partial_t \mathbf{A} = -\mathbf{E}$),

$$\begin{aligned}
\partial_t E_i &= -B_{ij,j} - \sum_s a^s \bar{\rho}^s v_i^s , \\
\partial_t A_i &= -E_i , \\
\partial_t \bar{\rho}^s &= -(\bar{\rho}^s v_j^s)_{,j} , \\
\partial_t \eta^s &= -v_j^s \eta^s_{,j} , \\
\partial_t v_i^s &= -v_j^s v_{i,j}^s - \frac{1}{\rho^s} P_{,i} + a^s (v_j^s B_{ji} + E_i) .
\end{aligned} \qquad [24]$$

These equations are obtained by introducing the multicomponent, single-pressure approximations into the standard plasma physics model, discussed e.g. in Holm & Kupershmidt (1983).

The static Maxwell source equation

$$E_{i,i} = \sum_s a^s \bar{\rho}^s$$

is compatible with the flow and will remain satisfied if it is initially true. Just as in adiabatic multiphase hydrodynamics, the constraints $\Sigma_s \theta^s = 1$ and $P^s(\rho^s, \eta^s) = P$ for all s impose a known (implicit) dependence

$$P = P(\{\bar\rho^s\}, \{\eta^s\}), \quad \theta^s = \theta^s(\{\bar\rho^s\}, \{\eta^s\})$$

for given functional relations

$$\rho^s = \rho^s(P, \eta^s) = \frac{\bar\rho^s}{\theta^s}$$

among the microscopic densities ρ^s, macroscopic densities $\bar\rho^s$, and volume fractions θ^s.

Hamiltonian formalism. Setting $\tilde{\mathbf{M}}^s = \bar\rho^s \mathbf{v}^s + a^s \bar\rho^s \mathbf{A}$, the MPP equations [24] can be cast into Hamiltonian form $\partial_t F = \{H, F\}$, with Poisson bracket given on the space of dynamical variables $\{\bar\rho^s, \eta^s, \tilde{\mathbf{M}}^s, \mathbf{E}, \mathbf{A}\}$ by

$$\begin{aligned} -\{K,J\} = \sum_s \int d^n x \Bigg\{ & \frac{\delta J}{\delta \bar\rho^s} \partial_j \bar\rho^s \frac{\delta K}{\delta \tilde M_j^s} + \frac{\delta J}{\delta \eta^s} \eta^s_{,j} \frac{\delta K}{\delta \tilde M_j^s} \\ & + \frac{\delta J}{\delta \tilde M_i^s} \left[\bar\rho^s \partial_i \frac{\delta K}{\delta \bar\rho^s} - \eta^s_{,i} \frac{\delta K}{\delta \eta^s} + (\partial_j \tilde M_i^s + \tilde M_j^s \partial_i) \frac{\delta K}{\delta \tilde M_j^s} \right] \Bigg\} \\ & + \int d^n x \left(\frac{\delta J}{\delta A_i} \frac{\delta K}{\delta E_i} - \frac{\delta K}{\delta A_i} \frac{\delta J}{\delta E_i} \right), \end{aligned} \qquad [25]$$

and Hamiltonian

$$H = \sum_s \int d^n x \left(\frac{|\tilde{\mathbf{M}}^s - a^s \bar\rho^s \mathbf{A}|^2}{2\bar\rho^s} + \bar\rho^s e^s(\rho^s, \eta^s) \right) + \int d^n x \left(\frac{1}{2} |\mathbf{E}|^2 + \frac{1}{2} B_{ij} B_{ij} \right). \qquad [26]$$

The variational derivatives

$$\begin{aligned} \frac{\delta H}{\delta \tilde M_j^s} &= v_j^s, \\ \frac{\delta H}{\delta \bar\rho^s} &= \frac{|\mathbf{v}^s|^2}{2} - a^s \mathbf{A}\cdot\mathbf{v}^s + e^s + \frac{P}{\rho^s}, \\ \frac{\delta H}{\delta \eta^s} &= \bar\rho^s T^s, \\ \frac{\delta H}{\delta A_i} &= -\sum_s a^s \bar\rho^s v_i^s - B_{ij,j}, \\ \frac{\delta H}{\delta E_i} &= E_i, \end{aligned}$$

readily imply the MPP equations [24] using [25] and [26].

Next, an invertible change of variables

$$M_i^s = \tilde M_i^s - a^s \bar\rho^s A_i = \bar\rho^s v_i^s$$

in the Poisson bracket [25], followed by noticing that the resulting bracket involves A_i only in the combination $A_{i,k} - A_{k,i} = B_{ik}$, leads to a gauge-invariant Poisson bracket in the space of magnetic fields plus physical variables

$$- \{K,J\} = \sum_s \int d^n x \left\{ \frac{\delta J}{\delta \bar{\rho}^s} \partial_j \bar{\rho}^s \frac{\delta K}{\delta M_j^s} + \frac{\delta J}{\delta \eta^s} \eta^s_{,j} \frac{\delta K}{\delta M_j^s} + \frac{\delta J}{\delta M_i^s} \left[\bar{\rho}^s \partial_i \frac{\delta K}{\delta \bar{\rho}^s} - \eta^s_{,i} \frac{\delta K}{\delta \eta^s} \right.\right.$$

$$\left.\left. + (\partial_j M_i^s + M_j^s \partial_i) \frac{\delta K}{\delta M_j^s} + a^s \bar{\rho}^s \frac{\delta K}{\delta E_i} + a^s \bar{\rho}^s B_{ji} \frac{\delta K}{\delta M_j^s} \right] - \frac{\delta J}{\delta E_i} a^s \bar{\rho}^s \frac{\delta H}{\delta M_i^s} \right\} \quad [27]$$

$$+ \int d^n x \left[\frac{\delta J}{\delta B_{ij}} \left(\partial_i \frac{\delta K}{\delta E_j} - \partial_j \frac{\delta K}{\delta E_i} \right) + \frac{\delta J}{\delta E_i} \partial_j \frac{\delta K}{\delta B_{ji}} \right].$$

Except for the presence of the macroscopic density $\bar{\rho}^s$ instead of the microscopic density ρ^s, the brackets [25] and [27] have the same form as those in Holm & Kupershmidt (1983) for the physically very different system of multifluid plasmas interacting via electromagnetic fields and multiple pressures, without imposing the constraint of pressure equilibration. For those multifluid plasmas, the bracket [27] in $\mathbb{R}^3$ was found in Iwinski & Turski (1976) and rediscovered in Kaufman & Spencer (1982).

To determine additional conservation laws for the MPP system, we first define a 1-form $\tilde{q}^s = (v_i^s + a^s A_i) dx^i$. Then by the MPP equations [24] one finds

$$(\partial_t + \mathscr{L}\mathbf{v}^s)\tilde{q}^s + \frac{1}{\rho^s} dP - d\left(\frac{|\mathbf{v}^s|^2}{2} + a^s \mathbf{v}^s \cdot \mathbf{A} \right) = 0 ,$$

where $\mathscr{L}\mathbf{v}^s$ is the Lie derivative with respect to $\mathbf{v}^s$. In addition, since $(\partial_t + \mathscr{L}\mathbf{v}^s) d\eta^s = 0$, we obtain, by proceeding as in section 3,

$$(\partial_t + \mathscr{L}\mathbf{v}^s)(d\tilde{q}^s \wedge d\eta^s) = dP \wedge d\left(\frac{1}{\rho^s}\right) \wedge d\eta^s = 0 ,$$

using $\rho^s = \rho^s(P, \eta^s)$ in the last equality. Thus, in three dimensions we have a plasma analog of [19] for adiabatic fluids;

$$(\partial_t + \mathbf{v}^s \cdot \nabla)\tilde{Q}^s = 0 ,$$

where

$$\tilde{Q}^s = (\bar{\rho}^s)^{-1} (\boldsymbol{\omega}^s + a^s \mathbf{B}) \cdot \nabla \eta^s$$

for each species, and $\mathbf{B} = \operatorname{curl} \mathbf{A}$ is the magnetic field vector.

Associated conserved quantities are

$$F_{\Phi^s} = \int d^3 x \, \bar{\rho}^s \, \Phi^s(\eta^s, \tilde{Q}^s) , \quad [28]$$

for arbitrary functions Φ^s of two variables. Again, the conserved quantities [28] associated with advection of $\tilde{Q}^s$ and η^s lie in the kernel of either Poisson bracket [25] or [27]; so they will be conserved for any choice of Hamiltonian in these dynamical variables.

5. LYAPUNOV STABILITY ANALYSIS FOR THE SINGLE-PRESSURE MODEL

As will be shown below, equilibrium (i.e. steady-state flows) of the multiphase equations [18] are extremal points of the sum $H_F = H + \Sigma_s F_{\Phi^s}$ defined by [21] and [23']. Lyapunov stability of these equilibrium states can be investigated by studying the conditions for definiteness of $\delta^2 H_F$, the second variation of H_F evaluated at the equilibrium state. The quantity $\delta^2 H_F$ is preserved by the linearized equations, and is the Hamiltonian for the dynamics linearized around the equilibrium state (see Arbarbanel *et al* 1986, appendix C) When the equilibrium state satisfies conditions sufficient for this second variation to be definite in sign, then the quantity $\delta^2 H_F$ defines a conserved norm, in terms of which the linearized equations will be Lyapunov stable (that is, when the equilibrium state satisfies the conditions required to make $\delta^2 H_F$ definite in sign, then in terms of the conserved norm $\delta^2 H_F$, every perturbed state remains in some neighborhood of equilibrium under the linearized dynamics). More detailed discussions of how fluid dynamical stability results are obtainable by Lyapunov's method in the context of the Hamiltonian formalism appear in, e.g. Arnold (1965; 1969), Holm *et al.* (1983), Holm (1984), Holm *et al* (1985). and Abarbanel *et al.* (1984, 1986)

Here we show that dependence of the common pressure P on the entire set of macroscopic densities $\{\bar{\rho}^s\}$ causes the second variation $\delta^2 H_F$ to be indefinite in sign and, thus, prevents linearized Lyapunov stability from being establishable in the vicinity of the stationary flows that are equilibrium states of H_F. This result is consistent with the ill posedness of the single-pressure multiphase equations in one dimension, as discussed in Stewart & Wendroff (1984) and references therein.

For simplicity, we consider two-dimensional, barotropic, multiphase flow in the x–y plane. Calculations which are more complicated, but completely analogous to those to be illustrated are also possible for the other multiphase flows considered in this paper. For equilibrium planar barotropic flows $\bar{\rho}^s_e$, $\mathbf{v}^s_e$, we have

$$\operatorname{div} \bar{\rho}^s_e \mathbf{v}^s_e = 0\,, \tag{29}$$

$$\mathbf{v}^s_e \cdot \nabla\left(\frac{\omega^s_e}{\bar{\rho}^s_e}\right) = 0\,, \tag{30a}$$

$$\mathbf{v}^s_e \cdot \nabla\left(\frac{1}{2}\left(|\mathbf{v}_e|^2 + h^s(P_e) + \phi\right)\right) = 0\,, \tag{30b}$$

with $\omega^s_e = \hat{\mathbf{z}}\cdot\operatorname{curl} \mathbf{v}^s_e$, where $\hat{\mathbf{z}}$ is the unit vector normal to the plane. The relation [30b] follows, upon scalar multiplication by $\mathbf{v}^s_e$, from the equilibrium relation

$$\mathbf{v}^s_e \times \hat{\mathbf{z}}\omega^s_e = \nabla\left(\frac{1}{2}|\mathbf{v}^s_e|^2 + h^s(P_e) + \phi\right). \tag{31}$$

For both sets of relations [30a] and [30b] to hold in the plane, it suffices that there exist functions K^s for which

$$\frac{|\mathbf{v}^s_e|^2}{2} + h^s(P_e) + \phi = K^s\left(\frac{\omega^s_e}{\bar{\rho}^s_e}\right), \tag{32}$$

provided $\mathbf{v}^s_e$ and ω^s_e are nonzero throughout the domain of flow considered. Then, taking the vector product of [31] with $\hat{\mathbf{z}}$ using [32] leads to

$$\bar{\rho}^s_e \mathbf{v}^s_e = \frac{K'^s(\omega^s_e/\bar{\rho}^s_e)}{\omega^s_e/\bar{\rho}^s_e}\,\hat{\mathbf{z}} \times \nabla\left(\frac{\omega^s_e}{\bar{\rho}^s_e}\right), \tag{33}$$

where prime in K'^s refers to the derivative of $K^s(\omega^s_e/\bar{\rho}^s_e)$ with respect to the indicated argument. If, further $(\nabla(\omega^s_e/\bar{\rho}^s_e)$ vanishes nowhere in the domain considered, then one easily sees that

$$\frac{K'^s(\omega^s_e/\bar{\rho}^s_e)}{\omega^s_e/\bar{\rho}^s_e} = \frac{\bar{\rho}^s_e \mathbf{v}^s_e \cdot \hat{\mathbf{z}} \times \nabla \omega^s_e/\bar{\rho}^s_e}{|\nabla \omega^s_e/\bar{\rho}^s_e|^2}, \tag{34}$$

upon scalar multiplying [33] by the quantity $\hat{\mathbf{z}} \times \nabla(\omega^s_e/\bar{\rho}^s_e)$.

We now show that the first variation

$$\delta H_F := \sum_s DH_F(\rho^s_e, \mathbf{v}^s_e) \cdot (\delta\rho^s, \delta\mathbf{v}^s) \tag{35}$$

vanishes for equilibrium flows ρ^s_e, $\mathbf{v}^s_e$, taking place in a domain D of the x–y plane with boundary conditions $\mathbf{v}^s \cdot \hat{\mathbf{n}} = 0$, where $\hat{\mathbf{n}}$ is the unit vector normal to the boundary ∂D. From [21] and [23′] we have

$$H_F = \sum_s \int_D \mathrm{d}x\mathrm{d}y \left[\frac{1}{2}\bar{\rho}^s|\mathbf{v}^s|^2 + \bar{\rho}^s e^s(\rho^s) + \bar{\rho}^s\phi + \bar{\rho}^s\Phi^s\left(\frac{\omega^s}{\bar{\rho}^s}\right)\right] + \sum_s \lambda^s \int_D \mathrm{d}x\mathrm{d}y\, \omega^s, \qquad [36]$$

where λ^s is a constant multiplying a term separated from Φ^s for convenience later in [39]. Using [2] and integrating by parts in [36] gives the expression

$$\delta H_F = \sum_s \int_D \mathrm{d}x\mathrm{d}y \left[\left(\frac{1}{2}|\mathbf{v}^s|^2 + e^s + \frac{P}{\rho^s} + \phi + \Phi^s - \frac{\omega^s}{\bar{\rho}^s}\Phi'^s\right)\delta\bar{\rho}^s + \left(\bar{\rho}^s\mathbf{v}^s - \Phi''^s\left(\frac{\omega^s}{\bar{\rho}^s}\right)\hat{\mathbf{z}} \times \nabla\frac{\omega^s}{\bar{\rho}^s}\right)\cdot\delta\mathbf{v}^s\right] + \sum_s \int_{\partial D}\left[\lambda^s + \Phi'^s\left(\frac{\omega^s}{\bar{\rho}^s}\right)\right]\delta\mathbf{v}^s \cdot \mathrm{d}\mathbf{l}. \qquad [37]$$

The first variation δH_F thus vanishes for equilibrium flows, provided the functions Φ^s satisfy the relations needed for each coefficient in [37] to vanish,

$$\frac{1}{2}|\mathbf{v}^s_e|^2 + h^s(P_e) + \phi = -\Phi^s\left(\frac{\omega^s_e}{\bar{\rho}^s_e}\right) + \frac{\omega^s_e}{\bar{\rho}^s_e}\Phi'^s\left(\frac{\omega^s_e}{\bar{\rho}^s_e}\right),\ \bar{\rho}^s_e\mathbf{v}^s_e = \Phi''^s\left(\frac{\omega^s_e}{\bar{\rho}^s_e}\right)\hat{\mathbf{z}} \times \nabla\left(\frac{\omega^s_e}{\bar{\rho}^s_e}\right) \qquad [38]$$

in the interior of domain D, and

$$\lambda^s + \Phi'^s\left(\frac{\omega^s_e}{\bar{\rho}^s_e}\right) = 0 \qquad [39]$$

on the boundary ∂D. The latter condition [39] is easily satisfied, since $\omega^s_e/\bar{\rho}^s_e$ is a constant on the boundary by [30] and the boundary condition $\mathbf{v}^s_e \cdot \hat{\mathbf{n}} = 0$. For both conditions in [38] to be satisfied, it suffices that

$$K^s\left(\frac{\omega^s_e}{\bar{\rho}^s_e}\right) = -\Phi^s\left(\frac{\omega^s_e}{\bar{\rho}^s}\right) + \frac{\omega^s_e}{\bar{\rho}^s_e}\Phi'^s\left(\frac{\omega^s_e}{\bar{\rho}^s}\right), \qquad [40]$$

so that the first relation in [38] holds. As a consequence, then

$$K'^s\left(\frac{\omega^s_e}{\bar{\rho}^s_e}\right) = \frac{\omega^s_e}{\bar{\rho}^s_e}\Phi''^s\left(\frac{\omega^s_e}{\bar{\rho}^s_e}\right), \qquad [41]$$

and the second relation in [38] is satisfied by virtue of [33] for steady flows. Therefore, H_F in [36] has a critical point for stationary flows, where the function Φ^s is determined by [40] to be

$$\Phi^s(q) = q\left(\int^q \frac{K^s(r)}{r^2}\,\mathrm{d}r + \text{const}\right). \qquad [42]$$

The second variation of H_F evaluated at equilibrium,

$$\delta^2 H_F = D^2 H_F(\bar{\rho}^s_e, \mathbf{v}^s_e)\cdot(\delta\bar{\rho}^s, \delta\mathbf{v}^s)^2, \qquad [43]$$

is given by

$$\delta^2 H_F = \sum_s \int dx\,dy \left\{ \bar{\rho}_e^s |\delta \mathbf{v}^s + \frac{\mathbf{v}_e^s}{\bar{\rho}_e^s} \delta\bar{\rho}^s|^2 + \Phi''^s \left(\frac{\omega_e^s}{\bar{\rho}_e^s}\right)\left(\delta\left(\frac{\omega^s}{\bar{\rho}^s}\right)\right)^2 \right.$$
$$\left. + \frac{1}{\bar{\rho}_e^s} \delta P_e(\{\bar{\rho}^\alpha\})\delta\bar{\rho}^\alpha - \frac{|\mathbf{v}_e^s|^2}{\bar{\rho}_e^s} (\delta\bar{\rho}^s)^2 \right\} \qquad [44]$$

If this expression could be made positive definite, the resulting conclusion would be that the linearized equations at equilibrium $\bar{\rho}_e^s$, $\mathbf{v}_e^s$ were Lyapunov stable in the preserved norm $\delta^2 H_F$. In the single-species case, this would require (cf. Holm *et al.* 1983)

$$\Phi''\left(\frac{\omega_e}{\rho_e}\right) = \frac{\rho_e \mathbf{v}_e \cdot \hat{\mathbf{z}} \times \nabla \omega_e/\rho_e}{|\nabla(\omega_e/\rho_e)|^2} > 0\,,$$

which is a compressible version of Rayleigh's inflection point criterion for stability (see e.g. Drazin & Reid 1981), and the condition

$$\frac{1}{\rho_e}\left[\frac{dP_e}{d\rho_e} - |\mathbf{v}_e|^2\right] > 0\,, \qquad [45]$$

which is the condition that the equilibrium flow be everywhere subsonic. However, for a multiphase flow one finds (cf. [1])

$$\delta P_e = \sum_{\beta=1}^{N} (\bar{C}_e^\beta)^2\, \delta\bar{\rho}^\beta\,, \qquad [46]$$

since the equilibrium pressure is a function of the entire set of macroscopic densities. Therefore, although $\delta^2 H_F$ is preserved by the linearized equations for multiphase flows, its preservation does not imply Lyapunov stability even for planar flows without inflection points, since $\delta^2 H_F$ is not definite in sign for such multiphase flows. The cause of this indefiniteness is the dependence of the common pressure at equilibrium P_e on the entire set of macroscopic densities $\{\bar{\rho}^s\}$. Thus, the constraint of common, instantaneously equilibrated pressures causes a difficulty, which we believe is unphysical (some equilibria must be stable!). This difficulty can be circumvented and Lyapunov stability obtained by formulating a very different multipressure theory of multiphase flows, as is discussed in the companion paper (Holm & Kupershmidt 1986).

6 CONCLUSION

We have presented Poisson brackets for multiphase hydrodynamics and electrohydrodynamics of ideal fluids and plasmas. In terms of macroscopic mass densities and momentum densities, these Poisson brackets have the same form, and retain the same mathematical structure as for the corresponding single-phase fluids. The kernels of these Poisson brackets give conservation laws that are independent of the choice of Hamiltonian for a particular theory. Equilibrium multiphase flows have been shown to be critical points of H_F, the energy constrained by these additional conservation laws. Linearized Lyapunov stability analyses of multiphase fluids have been given within the Hamiltonian context. Although the second variation $\delta^2 H_F$ of the constrained energy H_F is preserved by the linearized multiphase equations, no corresponding stability result can be concluded for the single-pressure multiphase case, since the dependence there of the pressure on all of the macroscopic densities causes $\delta^2 H_F$ to be indefinite in sign and, thus, not a stability norm. This difficulty can be resolved within the Hamiltonian framework by introducing multiple pressures, as is discussed in the companion paper (Holm & Kupershmidt 1986).

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